

Decompositions of 3-uniform hypergraph $K_v^{(3)}$ into hypergraph $K_4^{(3)} + e$ *

Tao Feng, Yanxun Chang
Institute of Mathematics
Beijing Jiaotong University
Beijing 100044, P. R. China
tfeng@bjtu.edu.cn
yxchang@bjtu.edu.cn

Abstract: In this paper it is established that a decomposition of a 3-uniform hypergraph $K_v^{(3)}$ into a special kind of hypergraph $K_4^{(3)} + e$ exists if and only if $v \equiv 0, 1, 2 \pmod{5}$ and $v \geq 7$.

Keywords: hypergraph decomposition; t -GDD; group divisible (Γ, t) -design; candelabra (Γ, t) -system

1 Introduction

A *hypergraph* H is a pair (V, E) , where V is a finite set of vertices, E is a family of subsets of V (called *hyperedges* or *edges*). A hypergraph is called *simple* if E has no repeated edges. All hypergraphs considered in this paper are simple. A *sub-hypergraph* $H' = (V', E')$ of $H = (V, E)$ is a hypergraph satisfying $V' \subseteq V$ and $E' \subseteq E$.

A hypergraph is said to be *t -uniform* if each of its edges contains exactly t vertices. In particular a 2-uniform hypergraph is just a graph. For a t -uniform hypergraph H , let $V(H)$ and $E(H)$ denote the vertex-set and edge-set of H , respectively. We say that H contains a vertex x if $x \in V(H)$, and H contains a set of vertices $\{x_1, x_2, \dots, x_t\}$ if $\{x_1, x_2, \dots, x_t\} \in E(H)$. A t -uniform hypergraph is said to be *complete* if the edge-set E contains each t -subset of V exactly once.

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It is denoted by $K_v^{(t)}$, where $v = |V|$ is called the *order* of the t -uniform hypergraph. The *degree* of a vertex x in a hypergraph is the number of edges that contain this vertex. It is denoted by $d(x)$. For more information on hypergraphs, the reader may refer to [1].

Let H be a t -uniform hypergraph and Γ be a set of t -uniform hypergraphs. A *decomposition* of H into hypergraphs of Γ is a partition of the edges of H into sub-hypergraphs each of which is isomorphic to a hypergraph in Γ . Such a decomposition of H into Γ is denoted by (H, Γ) -design. Hypergraph decompositions have an interesting application in secret sharing schemes (cf. [4]). When $H = K_v^{(t)}$, a $(K_v^{(t)}, \Gamma)$ -design is called a *t -wise balanced Γ design*, denoted by $S(t, \Gamma, v)$. If Γ only contains one hypergraph J , we write $S(t, \{J\}, v)$ simply as $S(t, J, v)$. Let K be a set of positive integers and Ω a set of complete t -uniform hypergraphs, where the order of each element in Ω is from K . We denote an $S(t, \Omega, v)$ by $S(t, K, v)$, which corresponds to the traditional concept of t -wise balanced design (t -BD) [2]. Therefore the t -wise balanced Γ design is a generalization of the t -wise balanced design.

One of the interesting problems in design theory is to determine the existence spectrum of $S(t, \Gamma, v)$, where Γ is a set of t -uniform hypergraphs. When $t = 2$, significant progress was made on this problem by many authors (e.g., see [3] and the references therein). However much less is known about $t \geq 3$. For $t = 3$, the necessary conditions for the existence of an $S(3, \Gamma, v)$ are as follows.

Lemma 1.1 ([5]) *Let Γ be any set of 3-uniform hypergraphs and $J \in \Gamma$. Let $d_J^*(x_1, x_2)$ be the number of edges in J containing the two vertices x_1 and x_2 . The following are necessary conditions for the existence of an $S(3, \Gamma, v)$:*

- (1) $v \geq \min\{|V(J)| : J \in \Gamma\};$
- (2) $\binom{v}{3} \equiv 0 \pmod{d_0}$, where $d_0 = \gcd\{|E(J)| : J \in \Gamma\};$
- (3) $\binom{v-1}{2} \equiv 0 \pmod{d_1}$, where $d_1 = \gcd\{d(x) : x \in V(J), J \in \Gamma\};$
- (4) $\binom{v-2}{1} \equiv 0 \pmod{d_2}$, where $d_2 = \gcd\{d_J^*(x_1, x_2) : x_1 \neq x_2, x_1, x_2 \in V(J), J \in \Gamma\}.$

For an edge $e \in E(K_4^{(3)})$, let $K_4^{(3)} - e$ denote the hypergraph obtained from $K_4^{(3)}$ by deleting the edge e . In [5], the authors investigated the existence of an $S(3, K_4^{(3)} - e, v)$ as follows.

Theorem 1.2 ([5]) *An $S(3, K_4^{(3)} - e, v)$ exists if and only if $v \equiv 0, 1, 2 \pmod{9}$ and $v \geq 9$.*

In [6] Hanani gave the following theorem.

Theorem 1.3 ([6]) *An $S(3, K_4^{(3)}, v)$ exists if and only if $v \equiv 2, 4 \pmod{6}$ and $v \geq 4$.*

Combining Theorems 1.2 and 1.3, it is natural to consider what the existence spectrum for an $S(3, K_4^{(3)} + e, v)$ is, where $K_4^{(3)} + e$ is a hypergraph (V, E) with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{3, 4, 5\}\}$. In this paper we establish that

Theorem 1.4 *An $S(3, K_4^{(3)} + e, v)$ exists if and only if $v \equiv 0, 1, 2 \pmod{5}$ and $v \geq 7$.*

For convenience, in this paper we always assume that K is a set of positive integers, Γ is a set of t -uniform hypergraphs, and Ω is a set of complete t -uniform hypergraphs, where the order of each element in Ω is from K .

2 Recursive constructions

To describe our recursive constructions, we need the following auxiliary designs. For the general background on design theory, the reader is referred to [2].

Let n and t be positive integers. Suppose that X is a set of points, \mathcal{B} is a collection of hypergraphs on the subsets of X (called *blocks*), and \mathcal{G} is a partition of X into n non-empty subsets (called *groups* or *holes*). A *group divisible (Γ, t) -design* is a triple $(X, \mathcal{G}, \mathcal{B})$, where for each $B \in \mathcal{B}$, B is isomorphic to a hypergraph in Γ , such that each edge from the edge-set of each block intersects any given group in at most one point, and each t -subset of X from t distinct groups is contained in a unique block.

We use the usual exponential notation for the types of group divisible (Γ, t) -designs. Then type $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$ denotes that there are a_i groups of size g_i , $1 \leq i \leq m$. For brevity, a group divisible (Γ, t) -design of type $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$ can be denoted by $\text{GDD}(t, \Gamma, v)$ of type $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$, where $v = \sum_{i=1}^m a_i g_i$. If Γ contains only one hypergraph J , we write $\text{GDD}(t, \{J\}, v)$ as $\text{GDD}(t, J, v)$.

If we replace Γ by Ω , then a $\text{GDD}(t, \Omega, v)$ is denoted by $\text{GDD}(t, K, v)$, which corresponds to the traditional concept of group divisible t -design (t -GDD) [11]. Therefore the group divisible (Γ, t) -design is a generalization of the group divisible t -design. Furthermore, if all the n groups have the same size g , a $\text{GDD}(t, K, v)$ is called an H design [10], denoted by $H(n, g, K, t)$.

Lemma 2.1 ([9]) *For any integer $4 \leq n \leq 27$ and $n \neq 5, 21$, there is a $\text{GDD}(3, \{4, 6\}, 2n)$ of type 2^n .*

The following construction is a variation of the fundamental construction for t -GDD [11].

Construction 2.2 ([5]) *Suppose that there exists a $\text{GDD}(t, K, v)$ of type $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$. If there exists a $\text{GDD}(t, \Gamma, hk)$ of type h^k for each $k \in K$, then there exists a $\text{GDD}(t, \Gamma, hv)$ of type $(hg_1)^{a_1} (hg_2)^{a_2} \cdots (hg_m)^{a_m}$.*

Let v , m and t be positive integers, and s be a non-negative integer. Suppose that X is a set of $v = s + \sum_{1 \leq i \leq m} a_i g_i$ points, S is a subset of X of size s (called *stem*), $\mathcal{T} = \{G_1, G_2, \dots, G_n\}$ is a partition of $X \setminus S$ of type $g_1^{a_1} \cdots g_m^{a_m}$, $n = \sum_{i=1}^m a_i$, (called *groups* or *branches*), and \mathcal{A} is a collection of hypergraphs on the subsets of X (called *blocks*). A *candelabra (Γ, t) -system* is a quadruple $(X, S, \mathcal{T}, \mathcal{A})$ of type $(g_1^{a_1} \cdots g_m^{a_m} : s)$, where for each $A \in \mathcal{A}$, A is isomorphic to one of Γ , such that every t -subset $T \subset X$ with $|T \cap (S \cup G_i)| < t$ for all i is contained in a unique block and no t -subset of $S \cup G_i$ is contained in any block. Such a system is denoted by $\text{CS}(t, \Gamma, v)$ of type $(g_1^{a_1} \cdots g_m^{a_m} : s)$. If Γ contains only one hypergraph J , we write $\text{CS}(t, \{J\}, v)$ as $\text{CS}(t, J, v)$.

If we replace Γ by Ω , then a $\text{CS}(t, \Gamma, v)$ is denoted by $\text{CS}(t, K, v)$, which corresponds to the traditional concept of candelabra t -design

[11]. Thus the candelabra (Γ, t) -system is a generalization of the candelabra t -system.

Lemma 2.3 *There exists a $CS(3, \{4, 6\}, 2n + 2)$ of type $(2^n : 2)$ for any integer $n \geq 3$.*

Proof For $n \equiv 0, 1 \pmod{3}$ and $n \geq 3$, let (X, \mathcal{B}) be an $S(3, 4, 2(n + 1))$ by Theorem 1.3. Let a and b be two distinct points in X . Let $\mathcal{T} = \{B \setminus \{a, b\} : B \in \mathcal{B}, \{a, b\} \subset B\}$ and $\mathcal{D} = \{B : B \in \mathcal{B}, \{a, b\} \subset B\}$. Then it is readily checked that $(X, \{a, b\}, \mathcal{T}, \mathcal{B} \setminus \mathcal{D})$ is a $CS(3, 4, 2n + 2)$ of type $(2^n : 2)$.

For $n \equiv 2 \pmod{3}$ and $n \geq 5$, let $(X', \emptyset, \mathcal{G}, \mathcal{B}')$ be a $CS(3, 4, 2(n + 1))$ of type $(6^{(n+1)/3} : 0)$ by Theorem 1 in [10]. Let c and d be two distinct points from two distinct groups in \mathcal{G} . Let $\mathcal{T}' = \{B \setminus \{c, d\} : B \in \mathcal{B}', \{c, d\} \subset B\}$ and $\mathcal{D}' = \{B : B \in \mathcal{B}', \{c, d\} \subset B\}$. Then it is readily checked that $(X', \{c, d\}, \mathcal{T}', (\mathcal{B}' \setminus \mathcal{D}') \cup \mathcal{G})$ is a $CS(3, \{4, 6\}, 2n + 2)$ of type $(2^n : 2)$. This completes the proof. \square

By Lemma 2.3, the following lemma is straightforward.

Lemma 2.4 ([7]) *There exists an $S(3, \{4, 6\}, v)$ for any integer $v \equiv 0 \pmod{2}$ and $v \geq 4$.*

We quote the following result for later use.

Lemma 2.5 ([7, 8]) *There exists an $S(3, \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27\}, v)$ for any integer $v \geq 4$.*

Construction 2.6 ([5]) *Suppose that there exists a $GDD(3, \Gamma, gn)$ of type g^n . If there is a $CS(3, \Gamma, 2g + s)$ of type $(g^2 : s)$, then there is a $CS(3, \Gamma, gn + s)$ of type $(g^n : s)$.*

Let $s \leq v$ be a non-negative integer. For convenience, in what follows a $(K_v^{(t)} \setminus K_s^{(t)}, \Gamma)$ -design is denoted by $HS(t, \Gamma; v, s)$, where the set of the s points is called the *hole* of this design. The following construction is simple but useful.

Construction 2.7 *Suppose that there exists a $CS(t, \Gamma, v)$ of type $(g_1^{a_1} \cdots g_{m-1}^{a_{m-1}} g_m^1 : s)$. If there is an $HS(t, \Gamma; g_i + s, s)$ for each $1 \leq i \leq m - 1$ and there is an $S(t, \Gamma, g_m + s)$, then there is an $S(t, \Gamma, v)$.*

Construction 2.8 Let $(X, S, \mathcal{G}, \mathcal{A})$ be a $CS(3, K, v)$ of type $(g_1^{a_1} \cdots g_m^{a_m} : s)$ with $S = \{x_1, x_2, \dots, x_s\}$. Suppose that for each block $A \in \mathcal{A}$ containing x_1 , there exists a $CS(3, \Gamma, b(|V(A)| - 1) + r)$ of type $(b^{|V(A)|-1} : r)$. Suppose that for each block $A \in \mathcal{A}$ containing x_i for $2 \leq i \leq s$, there exists a $GDD(3, \Gamma, b|V(A)|)$ of type $b^{|V(A)|}$. If for each block $A \in \mathcal{A}$ not containing x_i for any $1 \leq i \leq s$, there exists a $GDD(3, \Gamma, b|V(A)|)$ of type $b^{|V(A)|}$, then there exists a $CS(3, \Gamma, v')$ of type $((bg_1)^{a_1} \cdots (bg_m)^{a_m} : r + sb - b)$, where $v' = (v - 1)b + r$.

Proof For each block $A \in \mathcal{A}$ containing x_1 , construct a $CS(3, \Gamma, b(|V(A)| - 1) + r)$ of type $(b^{|V(A)|-1} : r)$ on $((V(A) \setminus \{x_1\}) \times Z_b) \cup S'$ with groups $\{x\} \times Z_b$, $x \in V(A) \setminus \{x_1\}$ and a stem S' of size r . We denote its block set by $\mathcal{B}_{A(x_1)}$. For each block $A \in \mathcal{A}$ containing x_i , $2 \leq i \leq s$, construct a $GDD(3, \Gamma, b|V(A)|)$ of type $b^{|V(A)|}$ on $V(A) \times Z_b$ with groups $\{x\} \times Z_b$, $x \in V(A)$. We denote its block set by $\mathcal{B}_{A(x_i)}$. For each block $A \in \mathcal{A}$ not containing x_i for any $1 \leq i \leq s$, construct a $GDD(3, \Gamma, b|V(A)|)$ of type $b^{|V(A)|}$ on $V(A) \times Z_b$ with groups $\{x\} \times Z_b : x \in V(A)\}$. We denote its block set by \mathcal{B}_A .

Let $X' = ((X \setminus \{x_1\}) \times Z_b) \cup S'$ and $\mathcal{G}' = \{G \times Z_b : G \in \mathcal{G}\}$. Let $S'' = (\cup_{i=2}^s (\{x_i\} \times Z_b)) \cup S'$. For $1 \leq i \leq s$, let \mathcal{B}_i denote the union of $\mathcal{B}_{A(x_i)}$ for all blocks $A \in \mathcal{A}$ containing x_i . Let \mathcal{B}' denote the union of \mathcal{B}_A for all blocks $A \in \mathcal{A}$ not containing x_i for any $1 \leq i \leq s$. Let $\mathcal{B} = (\cup_{i=1}^s \mathcal{B}_i) \cup \mathcal{B}'$. Then using similar arguments as in Construction 2.8 in [5], it is readily checked that $(X', S'', \mathcal{G}', \mathcal{B})$ is the required $CS(3, \Gamma, v')$. \square

Construction 2.8 in [5] can be seen as a corollary of the above construction.

3 Direct constructions

In the following we always denote the copy of $K_4^{(3)} + e$ with vertices x, y, z, u, v and edges $\{x, y, z\}$, $\{x, y, u\}$, $\{x, z, u\}$, $\{y, z, u\}$, $\{z, u, v\}$ by (x, y, z, u, v) .

Lemma 3.1 *There exists an $S(3, K_4^{(3)} + e, v)$ for $v \in \{7, 11, 16, 26, 31, 32\}$.*

Proof Let $X = Z_v$. Base blocks for these designs are given below. All other blocks are obtained by developing these base blocks by $+1$ modulo v .

$v = 7$:	(0, 1, 2, 4, 5).		
$v = 11$:	(0, 1, 2, 6, 4),	(0, 1, 3, 9, 7),	(0, 1, 4, 7, 8).
$v = 16$:	(0, 1, 2, 4, 5),	(0, 1, 5, 6, 12),	(0, 1, 8, 10, 3),
	(0, 1, 9, 13, 15),	(0, 2, 5, 10, 13),	(0, 2, 6, 13, 8),
	(0, 3, 6, 10, 15).		
$v = 26$:	(0, 1, 2, 4, 5),	(0, 1, 5, 6, 2),	(0, 1, 7, 8, 16),
	(0, 1, 10, 11, 22),	(0, 1, 13, 15, 2),	(0, 1, 14, 18, 2),
	(0, 2, 5, 7, 11),	(0, 2, 8, 10, 4),	(0, 2, 9, 12, 3),
	(0, 2, 11, 17, 3),	(0, 2, 16, 19, 3),	(0, 3, 6, 21, 10),
	(0, 3, 7, 11, 24),	(0, 3, 8, 15, 22),	(0, 3, 9, 14, 2),
	(0, 3, 16, 22, 4),	(0, 4, 9, 18, 2),	(0, 4, 10, 16, 3),
	(0, 4, 11, 21, 6),	(0, 5, 10, 18, 3).	
$v = 31$:	(0, 1, 2, 4, 5),	(0, 1, 5, 6, 2),	(0, 1, 7, 8, 16),
	(0, 1, 10, 11, 2),	(0, 1, 12, 13, 27),	(0, 1, 14, 17, 2),
	(0, 1, 16, 18, 2),	(0, 2, 5, 7, 11),	(0, 2, 8, 10, 4),
	(0, 2, 9, 11, 21),	(0, 2, 13, 19, 3),	(0, 2, 14, 20, 5),
	(0, 2, 18, 21, 3),	(0, 3, 6, 10, 2),	(0, 3, 8, 11, 2),
	(0, 3, 9, 14, 2),	(0, 3, 12, 21, 5),	(0, 3, 20, 24, 5),
	(0, 3, 22, 27, 17),	(0, 4, 9, 21, 15),	(0, 4, 10, 20, 3),
	(0, 4, 12, 19, 5),	(0, 4, 13, 17, 5),	(0, 4, 15, 24, 10),
	(0, 5, 11, 19, 28),	(0, 5, 13, 23, 30),	(0, 5, 15, 20, 2),
	(0, 6, 13, 24, 5),	(0, 6, 15, 23, 8).	
$v = 32$:	(0, 1, 2, 4, 5),	(0, 1, 5, 6, 2),	(0, 1, 7, 8, 16),
	(0, 1, 10, 11, 2),	(0, 1, 12, 13, 26),	(0, 1, 15, 16, 2),
	(0, 2, 5, 7, 11),	(0, 2, 8, 10, 4),	(0, 2, 9, 11, 21),
	(0, 2, 13, 15, 3),	(0, 2, 14, 16, 31),	(0, 3, 6, 10, 2),
	(0, 3, 8, 11, 2),	(0, 3, 9, 14, 2),	(0, 3, 12, 15, 2),
	(0, 3, 13, 17, 2),	(0, 3, 16, 21, 2),	(0, 3, 18, 25, 2),
	(0, 3, 19, 28, 2),	(0, 4, 9, 18, 5),	(0, 4, 10, 16, 29),
	(0, 4, 11, 25, 15),	(0, 4, 12, 20, 2),	(0, 4, 15, 27, 5),
	(0, 4, 17, 24, 5),	(0, 5, 10, 17, 4),	(0, 5, 11, 21, 3),
	(0, 5, 13, 22, 2),	(0, 5, 15, 24, 9),	(0, 6, 14, 20, 3),
	(0, 6, 17, 25, 4).		

□

Lemma 3.2 *There exists an $S(3, K_4^{(3)} + e, v)$ for $v \in \{10, 12, 15\}$.*

Proof For $v = 10$: let $X = Z_{10}$. All 24 blocks are listed below.

(0, 1, 8, 9, 5), (0, 1, 4, 5, 6), (0, 1, 6, 7, 4), (0, 2, 5, 8, 4),
 (0, 2, 7, 9, 5), (0, 4, 3, 9, 8), (0, 5, 3, 7, 4), (0, 3, 6, 8, 5),
 (0, 8, 4, 7, 9), (0, 6, 5, 9, 4), (4, 7, 1, 2, 0), (1, 2, 5, 9, 3),
 (1, 2, 6, 8, 7), (4, 8, 1, 3, 0), (1, 6, 3, 5, 8), (7, 9, 1, 3, 2),
 (1, 9, 4, 6, 8), (8, 1, 5, 7, 4), (4, 5, 2, 3, 0), (2, 9, 3, 6, 7),
 (2, 3, 7, 8, 9), (2, 4, 8, 9, 6), (0, 2, 4, 6, 3), (2, 5, 6, 7, 9).

For $v = 12$: let $X = Z_{11} \cup \{\infty\}$. Base blocks for this design are given below. All other blocks are obtained by developing these base blocks by $+1$ modulo 11, where $\infty + 1 = \infty$.

(8, 1, 0, 5, ∞), (0, 1, ∞ , 3, 7), (1, 4, 6, 0, 2), (7, 9, 1, 0, 2).

For $v = 15$: let $X = Z_{13} \cup \{\infty_1, \infty_2\}$. Base blocks for this design are given below. All other blocks are obtained by developing these base blocks by $+1$ modulo 13, where $\infty_i + 1 = \infty_i$ for $i = 1, 2$.

(0, 1, 4, ∞_1, ∞_2), (∞_1 , 7, 0, 2, 4), (∞_2 , 5, 0, 7, 1),
 (∞_2 , 10, 0, 1, 9), (2, 6, 0, 1, 3), (1, 11, 0, 8, 3),
 (2, 5, 0, 9, 3). □

Lemma 3.3 *There exists a $GDD(3, K_4^{(3)} + e, gn)$ of type g^n for $(g, n) \in \{(5, 4), (5, 6), (10, 5)\}$.*

Proof For $(g, n) = (5, 4)$: let $X = Z_{20}$ and $\mathcal{G} = \{4Z_5 + j : 0 \leq j \leq 3\}$. Base blocks for this design are given below. All other blocks are obtained by developing these base blocks by $+1$ modulo 20.

(0, 1, 2, 7, 12), (0, 1, 3, 14, 4), (0, 1, 15, 18, 5), (0, 3, 9, 18, 8),
 (0, 5, 7, 14, 17).

For $(g, n) = (5, 6)$: let $X = Z_{25} \cup \{\infty_0, \infty_1, \dots, \infty_4\}$ and $\mathcal{G} = \{5Z_5 + i : 0 \leq i \leq 4\} \cup \{\infty_0, \infty_1, \dots, \infty_4\}$. Base blocks for this design are given below. All other blocks are obtained by developing these base blocks by $+1$ modulo 25, where $\infty_j + 1 = \infty_{j+1}$, $0 \leq j \leq 4$, the subscripts are reduced modulo 5.

(∞_0 , 2, 0, 1, 9), (∞_0 , 3, 0, 4, 11), (∞_0 , 8, 0, 6, 18),
 (∞_0 , 9, 0, 7, 14), (∞_0 , 14, 0, 11, 8), (∞_0 , 12, 0, 13, 19),
 (0, 16, 22, ∞_0 , 1), (0, 17, 24, ∞_0 , 1), (0, 18, 21, ∞_0 , 3),
 (0, 19, 23, ∞_0 , 2), (4, 11, 23, ∞_0 , 7), (3, 14, 22, ∞_0 , 8),
 (1, 14, 17, ∞_0 , 4), (4, 13, 21, ∞_0 , 7), (0, 1, 4, 7, 5),
 (0, 1, 8, 12, 4), (0, 1, 14, 23, 2), (0, 2, 4, 13, 7),

$$(0, 2, 6, 14, 3), \quad (0, 6, 7, 23, 14).$$

For $(g, n) = (10, 5)$: let $X = Z_{50}$ and $\mathcal{G} = \{5Z_{10} + j : 0 \leq j \leq 4\}$. The 40 base blocks for this design can be obtained by multiplying each of the following 8 base blocks by $(11)^i$, $i = 0, 1, 2, 3, 4$. All other blocks are obtained by developing these base blocks by $+1$ modulo 50.

$$\begin{aligned} (0, 1, 2, 4, 5), & \quad (0, 1, 7, 8, 4), & \quad (0, 1, 9, 33, 5), \\ (0, 1, 13, 19, 5), & \quad (0, 1, 17, 23, 10), & \quad (0, 1, 18, 42, 5), \\ (0, 1, 27, 29, 3), & \quad (0, 1, 28, 34, 41). \end{aligned}$$

□

Lemma 3.4 *There exists a $CS(3, K_4^{(3)} + e, gn)$ of type $(g^n : 0)$ for $(g, n) \in \{(5, 3), (5, 5), (10, 2), (15, 2)\}$.*

Proof Let $X = Z_{gn}$, $S = \emptyset$ and $\mathcal{T} = \{nZ_g + j : 0 \leq j \leq n - 1\}$. Base blocks for these designs are given below.

For $(g, n) = (5, 3)$, develop the following base blocks by $+3$ modulo 15.

$$\begin{aligned} (0, 6, 1, 7, 14), & \quad (1, 8, 0, 11, 4), & \quad (0, 4, 2, 6, 11), & \quad (0, 5, 2, 7, 10), \\ (0, 8, 2, 9, 6), & \quad (0, 2, 10, 11, 5), & \quad (2, 5, 1, 4, 3), & \quad (1, 2, 0, 3, 5), \\ (1, 4, 0, 12, 11), & \quad (1, 5, 0, 14, 2), & \quad (3, 13, 0, 10, 14), & \quad (4, 13, 0, 5, 12), \\ (4, 10, 0, 8, 5), & \quad (0, 4, 9, 14, 3), & \quad (2, 7, 1, 9, 3), & \quad (2, 3, 10, 14, 4), \\ (1, 14, 4, 8, 6). \end{aligned}$$

For $(g, n) = (5, 5)$, develop the following base blocks by $+1$ modulo 25.

$$\begin{aligned} (0, 1, 2, 4, 5), & \quad (0, 1, 5, 6, 2), & \quad (0, 1, 7, 8, 16), & \quad (0, 1, 10, 12, 2), \\ (0, 1, 11, 14, 2), & \quad (0, 1, 13, 16, 5), & \quad (0, 1, 15, 17, 2), & \quad (0, 2, 5, 7, 11), \\ (0, 2, 8, 13, 20), & \quad (0, 2, 9, 19, 16), & \quad (0, 2, 14, 21, 23), & \quad (0, 3, 6, 10, 13), \\ (0, 3, 8, 19, 11), & \quad (0, 3, 9, 20, 24), & \quad (0, 4, 8, 16, 3), & \quad (0, 4, 9, 15, 3), \\ (0, 4, 10, 17, 6), & \quad (0, 4, 13, 20, 2). \end{aligned}$$

For $(g, n) = (10, 2)$, develop the following base blocks by $+1$ modulo 20.

$$\begin{aligned} (0, 1, 2, 5, 3), & \quad (0, 1, 6, 7, 14), & \quad (0, 1, 9, 16, 2), & \quad (0, 1, 10, 13, 7), \\ (0, 1, 11, 18, 14), & \quad (0, 1, 12, 17, 10), & \quad (0, 2, 9, 17, 12), & \quad (0, 2, 11, 15, 18), \\ (0, 3, 9, 14, 4). \end{aligned}$$

For $(g, n) = (15, 2)$, develop the following base blocks by $+1$ modulo 30.

$$\begin{aligned} (0, 1, 2, 5, 3), & \quad (0, 1, 6, 7, 2), & \quad (0, 1, 8, 9, 5), & \quad (0, 1, 10, 11, 8), \\ (0, 1, 12, 15, 2), & \quad (0, 1, 13, 18, 2), & \quad (0, 1, 14, 17, 5), & \quad (0, 1, 16, 19, 8), \\ (0, 2, 7, 9, 4), & \quad (0, 2, 11, 19, 16), & \quad (0, 2, 13, 17, 4), & \quad (0, 2, 15, 21, 10), \end{aligned}$$

(0, 3, 6, 11, 16), (0, 3, 7, 12, 19), (0, 3, 9, 26, 4), (0, 3, 10, 23, 15),
 (0, 3, 13, 24, 8), (0, 4, 11, 23, 2), (0, 4, 15, 25, 10), (0, 5, 14, 21, 7),
 (0, 6, 13, 21, 28).

□

Lemma 3.5 *There exists a $CS(3, K_4^{(3)} + e, gn + 1)$ of type $(g^n : 1)$ for $(g, n) \in \{(5, 3), (5, 5), (6, 4), (10, 2)\}$.*

Proof Let $X = Z_{gn} \cup \{\infty\}$, $S = \{\infty\}$ and $\mathcal{T} = \{nZ_g + j : 0 \leq j \leq n - 1\}$. Base blocks for these designs are given below.

For $(g, n) = (5, 3)$, develop the following base blocks by $+3$ modulo 15, where $\infty + 3 = \infty$.

($\infty, 2, 1, 0, 9$), ($0, 4, 8, \infty, 9$), ($0, 5, 10, \infty, 2$), ($0, 11, 13, \infty, 6$),
 ($0, 7, 1, 5, 10$), ($0, 11, 1, 8, 10$), ($0, 4, 2, 5, 13$), ($0, 1, 3, 4, 2$),
 ($0, 6, 1, 10, 2$), ($0, 7, 2, 3, 10$), ($2, 8, 0, 6, 5$), ($0, 12, 2, 10, 11$),
 ($0, 14, 2, 13, 3$), ($0, 3, 8, 14, 4$), ($0, 11, 3, 10, 13$), ($0, 6, 4, 13, 14$),
 ($0, 11, 4, 7, 8$), ($0, 5, 8, 9, 14$), ($1, 3, 11, 13, 8$), ($1, 11, 2, 4, 8$).

For $(g, n) = (5, 5)$, develop the following base blocks by $+1$ modulo 25, where $\infty + 1 = \infty$.

($1, 8, 0, 9, \infty$), ($10, 11, 0, 1, \infty$), ($1, 14, 0, 12, \infty$), ($1, 13, 0, 19, \infty$),
 ($1, 20, 0, 21, \infty$), ($1, 24, 0, 22, \infty$), ($2, 5, 0, 7, \infty$), ($2, 6, 0, 8, \infty$),
 ($2, 9, 0, 11, \infty$), ($10, 17, 0, 2, \infty$), ($0, 2, 12, 15, 13$), ($0, 3, 6, 10, 7$),
 ($0, 3, 8, 11, 2$), ($0, 3, 9, 18, 13$), ($0, 3, 12, 16, 2$), ($0, 3, 14, 21, 9$),
 ($0, 4, 8, 14, 7$), ($0, 4, 11, 17, 2$), ($0, 4, 12, 20, 7$), ($0, 5, 11, 16, 4$).

For $(g, n) = (6, 4)$, develop the following base blocks by $+1$ modulo 24, where $\infty + 1 = \infty$.

($2, 4, 0, 1, \infty$), ($1, 6, 0, 5, \infty$), ($0, 8, 1, 7, \infty$), ($1, 10, 0, 9, \infty$),
 ($0, 1, 11, 13, \infty$), ($1, 12, 0, 21, \infty$), ($0, 22, 1, 14, \infty$), ($2, 5, 0, 7, \infty$),
 ($0, 2, 6, 20, \infty$), ($0, 2, 8, 18, 6$), ($0, 2, 9, 11, 19$), ($0, 3, 6, 12, 21$),
 ($0, 3, 7, 20, 11$), ($0, 3, 8, 19, 11$), ($0, 3, 10, 13, 4$), ($0, 4, 9, 19, 5$),
 ($0, 5, 10, 17, 2$), ($0, 5, 11, 18, 2$).

For $(g, n) = (10, 2)$, develop the following base blocks by $+1$ modulo 20, where $\infty + 1 = \infty$.

($2, 9, 0, 1, \infty$), ($1, 12, 0, 3, \infty$), ($1, 4, 0, 7, \infty$), ($1, 16, 0, 5, \infty$),
 ($0, 1, 6, 15, \infty$), ($0, 1, 10, 17, 12$), ($0, 1, 11, 14, 16$), ($0, 1, 13, 18, 3$),
 ($0, 2, 5, 13, 8$), ($0, 3, 7, 14, 2$).

□

Lemma 3.6 *There exists a $CS(3, K_4^{(3)} + e, gn + 2)$ of type $(g^n : 2)$ for each $(g, n) \in \{(5, 3), (5, 5), (10, 2)\}$.*

Proof Let $X = Z_{gn} \cup \{\infty_1, \infty_2\}$, $S = \{\infty_1, \infty_2\}$ and $\mathcal{T} = \{nZ_g + j : 0 \leq j \leq n - 1\}$. Base blocks for these designs are given below.

For $(g, n) = (5, 3)$, develop the following base blocks by $+3$ modulo 15, where $\infty_i + 3 = \infty_i$ for $i = 1, 2$.

$(\infty_1, 1, 0, 2, 13), (0, 4, 8, \infty_1, 9), (0, 5, 10, \infty_1, 2), (0, 11, 13, \infty_1, 6),$
 $(\infty_2, 1, 0, 5, 13), (2, 9, 7, \infty_2, 8), (0, 4, 2, \infty_2, 6), (2, 10, 3, \infty_2, 13),$
 $(1, 8, 0, 10, 13), (1, 4, 0, 3, 11), (0, 7, 1, 9, 6), (0, 1, 11, 14, 6),$
 $(0, 5, 2, 3, 11), (0, 8, 2, 6, 14), (0, 2, 7, 10, 5), (0, 7, 3, 8, 10),$
 $(0, 14, 3, 13, 12), (6, 10, 0, 4, 14), (0, 4, 7, 11, 14), (0, 11, 5, 6, 12),$
 $(1, 5, 2, 4, 11), (1, 8, 2, 7, 14), (1, 5, 7, 11, 12).$

For $(g, n) = (5, 5)$, develop the following base blocks by $+1$ modulo 25, where $\infty_i + 1 = \infty_i$ for $i = 1, 2$.

$(0, 9, 1, 8, \infty_1), (\infty_1, 1, 3, 0, \infty_2), (\infty_1, 0, 13, 4, \infty_2), (\infty_1, 6, 14, 0, \infty_2),$
 $(0, 4, 17, \infty_2, 19), (\infty_2, 7, 1, 0, 6), (0, 1, 10, 11, 4), (0, 1, 12, 14, 2),$
 $(0, 1, 13, 20, 2), (0, 1, 21, 24, 23), (0, 2, 4, 7, 3), (0, 2, 6, 8, 15),$
 $(0, 2, 10, 17, 15), (0, 2, 11, 16, 5), (0, 2, 12, 18, 3), (0, 3, 6, 19, 23),$
 $(0, 3, 7, 11, 2), (0, 3, 8, 14, 5), (0, 3, 10, 13, 2), (0, 3, 12, 20, 7),$
 $(0, 4, 9, 16, 5), (0, 4, 10, 14, 9).$

For $(g, n) = (10, 2)$, develop the following base blocks by $+1$ modulo 20, where $\infty_i + 1 = \infty_i$ for $i = 1, 2$.

$(8, 9, 0, 1, \infty_1), (1, 18, 0, 3, \infty_1), (1, 4, 0, 5, \infty_1), (1, 6, 0, 7, \infty_1),$
 $(2, 7, 0, 9, \infty_1), (2, 11, 0, 1, \infty_2), (6, 13, 0, 3, \infty_2), (3, 16, 0, 7, \infty_2),$
 $(3, 8, 0, 15, \infty_2), (3, 12, 0, 9, \infty_2), (0, 15, 4, 9, 14).$

□

Lemma 3.7 *There exists an $HS(3, K_4^{(3)} + e; 15, 5)$.*

Proof The required design is constructed on $\{0, 1, \dots, 14\}$ with a hole $\{10, 11, 12, 13, 14\}$. All 89 blocks are listed below.

$(0, 9, 1, 8, 6), (0, 2, 1, 3, 13), (0, 5, 1, 4, 10), (0, 1, 6, 7, 9),$
 $(10, 11, 0, 1, 14), (1, 12, 0, 13, 2), (2, 4, 0, 6, 9), (0, 2, 5, 7, 10),$
 $(0, 2, 8, 10, 11), (0, 2, 9, 11, 10), (0, 2, 12, 14, 9), (3, 4, 0, 7, 8),$
 $(0, 6, 3, 5, 9), (8, 11, 0, 3, 12), (3, 9, 0, 10, 5), (0, 3, 13, 14, 8),$
 $(0, 4, 8, 12, 10), (0, 4, 9, 13, 12), (0, 10, 4, 14, 5), (0, 13, 5, 8, 2),$
 $(0, 9, 5, 12, 6), (0, 14, 5, 11, 7), (0, 14, 6, 8, 5), (0, 10, 6, 12, 7),$
 $(0, 13, 6, 11, 5), (0, 9, 7, 14, 5), (0, 10, 7, 13, 6), (0, 12, 7, 11, 6),$

$(2, 4, 1, 7, 9), (2, 6, 1, 5, 11), (1, 8, 2, 11, 6), (1, 2, 9, 10, 6),$
 $(1, 2, 13, 14, 5), (1, 4, 3, 6, 10), (1, 3, 5, 7, 9), (1, 3, 8, 10, 13),$
 $(1, 3, 9, 11, 5), (1, 3, 12, 14, 8), (1, 8, 4, 13, 6), (1, 9, 4, 12, 7),$
 $(1, 14, 4, 11, 0), (1, 12, 5, 8, 11), (1, 13, 5, 9, 6), (1, 10, 5, 14, 6),$
 $(1, 14, 6, 9, 8), (1, 10, 6, 13, 12), (1, 12, 6, 11, 10), (1, 8, 7, 14, 12),$
 $(1, 7, 10, 12, 9), (1, 11, 7, 13, 5), (2, 5, 3, 4, 8), (2, 6, 3, 7, 11),$
 $(2, 3, 8, 9, 14), (10, 11, 2, 3, 14), (3, 13, 2, 12, 1), (8, 14, 2, 4, 9),$
 $(4, 12, 2, 10, 7), (2, 4, 11, 13, 9), (2, 9, 5, 14, 12), (2, 13, 5, 10, 9),$
 $(2, 5, 11, 12, 9), (2, 6, 8, 12, 13), (2, 9, 6, 13, 5), (2, 6, 10, 14, 8),$
 $(2, 13, 7, 8, 10), (2, 12, 7, 9, 8), (2, 7, 11, 14, 8), (3, 4, 9, 14, 10),$
 $(3, 4, 10, 13, 9), (3, 4, 11, 12, 8), (3, 14, 5, 8, 7), (3, 12, 5, 10, 8),$
 $(3, 5, 11, 13, 8), (3, 6, 8, 13, 9), (3, 6, 9, 12, 8), (3, 6, 11, 14, 9),$
 $(3, 8, 7, 12, 13), (3, 13, 7, 9, 11), (3, 14, 7, 10, 11), (4, 5, 6, 7, 14),$
 $(4, 5, 8, 9, 11), (4, 11, 5, 10, 6), (4, 13, 5, 12, 7), (4, 8, 6, 10, 7),$
 $(4, 9, 6, 11, 8), (4, 12, 6, 14, 13), (4, 11, 7, 8, 6), (4, 7, 9, 10, 8),$
 $(4, 7, 13, 14, 9).$

□

Lemma 3.8 *There exists an $HS(3, K_4^{(3)} + e; 16, 6)$.*

Proof The required design is constructed on $\{0, 1, \dots, 15\}$ with a hole $\{10, 11, 12, 13, 14, 15\}$. All 108 blocks are listed below.

$(0, 12, 7, 11, 10), (1, 7, 2, 4, 3), (1, 2, 5, 6, 11), (1, 2, 8, 11, 5),$
 $(1, 2, 9, 10, 5), (1, 2, 12, 15, 8), (1, 2, 13, 14, 8), (1, 3, 4, 6, 15),$
 $(1, 7, 3, 5, 4), (1, 3, 8, 10, 5), (1, 3, 9, 11, 5), (1, 3, 12, 14, 5),$
 $(1, 3, 13, 15, 7), (1, 8, 4, 13, 7), (1, 9, 4, 12, 14), (1, 15, 4, 10, 7),$
 $(1, 11, 4, 14, 6), (1, 8, 5, 12, 7), (1, 13, 5, 9, 7), (1, 10, 5, 14, 6),$
 $(1, 11, 5, 15, 7), (1, 15, 6, 8, 7), (1, 14, 6, 9, 7), (1, 10, 6, 13, 7),$
 $(1, 11, 6, 12, 4), (1, 14, 7, 8, 5), (1, 15, 7, 9, 4), (1, 12, 7, 10, 6),$
 $(1, 11, 7, 13, 5), (2, 3, 6, 7, 11), (2, 3, 8, 9, 6), (2, 3, 10, 11, 6),$
 $(2, 3, 12, 13, 6), (2, 3, 14, 15, 8), (2, 8, 4, 14, 7), (2, 15, 4, 9, 10),$
 $(2, 10, 4, 12, 15), (2, 11, 4, 13, 15), (2, 15, 5, 8, 6), (2, 14, 5, 9, 6),$
 $(2, 10, 5, 13, 6), (2, 11, 5, 12, 6), (2, 8, 6, 12, 7), (2, 9, 6, 13, 14),$
 $(2, 14, 6, 10, 5), (2, 15, 6, 11, 8), (2, 8, 7, 13, 12), (2, 12, 7, 9, 10),$
 $(2, 10, 7, 15, 12), (2, 11, 7, 14, 5), (3, 4, 8, 15, 9), (3, 4, 9, 14, 8),$
 $(3, 10, 4, 13, 14), (0, 1, 14, 15, 7), (0, 2, 4, 6, 13), (0, 2, 5, 7, 10),$
 $(0, 2, 8, 10, 9), (0, 2, 9, 11, 8), (0, 2, 12, 14, 7), (0, 2, 13, 15, 6),$
 $(0, 3, 4, 7, 15), (0, 3, 5, 6, 15), (0, 3, 8, 11, 12), (0, 3, 9, 10, 6),$
 $(0, 3, 12, 15, 6), (0, 3, 13, 14, 5), (0, 4, 8, 12, 14), (0, 4, 9, 13, 14),$

$(0, 4, 10, 14, 9), (0, 4, 11, 15, 8), (0, 5, 8, 13, 11), (0, 5, 9, 12, 15),$
 $(0, 5, 10, 15, 8), (0, 5, 11, 14, 9), (0, 6, 8, 14, 11), (0, 6, 9, 15, 10),$
 $(0, 6, 10, 12, 8), (0, 6, 11, 13, 9), (0, 7, 8, 15, 13), (0, 7, 9, 14, 12),$
 $(0, 7, 10, 13, 8), (3, 4, 11, 12, 9), (3, 5, 8, 14, 10), (3, 5, 9, 15, 13),$
 $(3, 10, 5, 12, 15), (3, 11, 5, 13, 15), (3, 6, 8, 13, 9), (3, 6, 9, 12, 10),$
 $(3, 10, 6, 15, 14), (3, 11, 6, 14, 12), (3, 8, 7, 12, 4), (3, 7, 9, 13, 10),$
 $(3, 10, 7, 14, 13), (3, 15, 7, 11, 5), (4, 5, 6, 7, 14), (4, 5, 8, 9, 12),$
 $(4, 5, 10, 11, 9), (4, 5, 12, 13, 9), (4, 5, 14, 15, 9), (4, 6, 8, 10, 7),$
 $(4, 6, 9, 11, 15), (4, 8, 7, 11, 9), (0, 1, 2, 3, 5), (0, 1, 4, 5, 2),$
 $(0, 1, 6, 7, 15), (0, 1, 8, 9, 7), (0, 1, 10, 11, 8), (0, 1, 12, 13, 8).$

□

4 Conclusion

In this section we give the necessary and sufficient conditions for the existence of an $S(3, K_4^{(3)} + e, v)$.

Lemma 4.1 *There does not exist an $S(3, K_4^{(3)} + e, v)$ for $v = 5, 6$.*

Proof Let (X, \mathcal{B}) be an $S(3, K_4^{(3)} + e, v)$ for $v = 5, 6$. For any $x \in X$ denote by $d_i(x)$, $i = 1, 3, 4$, the number of blocks in \mathcal{B} in which the degree of x is i . It follows that $d_1(x) + 3d_3(x) + 4d_4(x) = (v - 1)(v - 2)/2$. Note that $d_1(x) + d_3(x) + d_4(x) \leq |\mathcal{B}|$.

For $v = 5$, solving the equation with the constraint $|\mathcal{B}| = 2$, we have that for any $x \in X$, $d_1(x) = d_4(x) = 0$ and $d_3(x) = 2$. It is easy to see that it is impossible. That is a contradiction.

For $v = 6$, solving the equation with the constraint $|\mathcal{B}| = 4$, we have that:

- (1) $d_1(x) = 0$, $d_3(x) = 2$ and $d_4(x) = 1$ (such vertex x is called a-vertex);
- (2) $d_1(x) = 1$, $d_3(x) = 3$ and $d_4(x) = 0$ (such vertex x is called b-vertex);
- (3) $d_1(x) = 2$, $d_3(x) = 0$ and $d_4(x) = 2$ (such vertex x is called c-vertex).

Denote by α, β, γ the number of a-vertices, b-vertices and c-vertices respectively. Since each block of \mathcal{B} contains exactly one vertex with degree 1, we have that $\beta + 2\gamma = |\mathcal{B}| = 4$. Since each block of \mathcal{B}

contains exactly two vertices with degree 3, we have that $2\alpha + 3\beta = 2|\mathcal{B}| = 8$. Due to $\alpha + \beta + \gamma = 6$, we have that $\alpha = 4$, $\beta = 0$ and $\gamma = 2$. Let $X = \{1, 2, \dots, 6\}$, in which 1, 2 are c-vertices and 3, 4, 5, 6 are a-vertices. Without loss of generality we can assume that the 4 blocks of \mathcal{B} are $(5, 6, 2, 3, 1)$, $(3, r, 2, 4, 1)$, $(3, *, 1, 5, 2)$, $(*, *, 1, 6, 2)$ or $(*, *, 2, 3, 1)$, $(*, *, 2, 4, 1)$, $(3, 4, 1, 5, 2)$, $(3, r, 1, 6, 2)$. Obviously any r from $\{1, 2, \dots, 6\}$ is impossible. That is a contradiction. \square

Lemma 4.2 *There exists an $S(3, K_4^{(3)} + e, v)$ for $v \equiv 0, 1, 2 \pmod{10}$.*

Proof By Lemmas 3.1 and 3.2, there exists an $S(3, K_4^{(3)} + e, v)$ for $v = 10, 11, 12, 31, 32$. There exists a $CS(3, K_4^{(3)} + e, 30)$ of type $(15^2 : 0)$ from Lemma 3.4. Then apply Construction 2.7 with an $S(3, K_4^{(3)} + e, 15)$ from Lemma 3.2 to obtain an $S(3, K_4^{(3)} + e, 30)$.

By Lemma 2.1, there exists a $GDD(3, \{4, 6\}, 2n)$ of type 2^n for any integer $4 \leq n \leq 27$ and $n \neq 5, 21$. Then apply Construction 2.2 with a $GDD(3, K_4^{(3)} + e, 5k)$ of type 5^k for $k = 4, 6$ from Lemma 3.3 to obtain a $GDD(3, K_4^{(3)} + e, 10n)$ of type 10^n .

By Lemma 2.5 there exists an $S(3, \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27\}, u)$ for any integer $u \geq 4$, which implies the existence of a $GDD(3, \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27\}, u)$ of type 1^u . Then apply Construction 2.2 to obtain a $GDD(3, K_4^{(3)} + e, 10u)$ of type 10^u , where the needed $GDD(3, K_4^{(3)} + e, 50)$ of type 10^5 is from Lemma 3.3, and the needed $GDD(3, K_4^{(3)} + e, 10m)$ of type 10^m for $m = 4, 6, 7, 9, 11, 13, 15, 19, 23, 27$ have been constructed in the second paragraph.

Start with the resulting $GDD(3, K_4^{(3)} + e, 10u)$ of type 10^u , and apply Construction 2.6 to obtain a $CS(3, K_4^{(3)} + e, 10u + s)$ of type $(10^u : s)$ for $u \geq 2$, $u \neq 3$ and $s = 0, 1, 2$, where the needed $CS(3, K_4^{(3)} + e, 20 + s)$ of type $(10^2 : s)$ are from Lemmas 3.4, 3.5 and 3.6. Fill in holes by Construction 2.7 to obtain an $S(3, K_4^{(3)} + e, 10u + s)$. \square

Lemma 4.3 *There exists an $S(3, K_4^{(3)} + e, v)$ for $v \equiv 5, 6 \pmod{10}$ and $v \geq 15$.*

Proof By Lemmas 3.1 and 3.2, there exists an $S(3, K_4^{(3)} + e, v)$ for $v = 15, 16, 26$. There exists a $CS(3, K_4^{(3)} + e, 25)$ of type $(6^4 : 1)$ from Lemma 3.5. Then apply Construction 2.7 with an $S(3, K_4^{(3)} + e, 7)$ from Lemma 3.1 to obtain an $S(3, K_4^{(3)} + e, 25)$.

By Lemma 2.3, there exists a $CS(3, \{4, 6\}, 2u + 2)$ of type $(2^u : 2)$ for any integer $u \geq 3$. Then applying Construction 2.8 with $b = 5$ and $r = 0, 1$, we obtain a $CS(3, K_4^{(3)} + e, 10u + 5 + r)$ of type $(10^u : 5 + r)$, where the needed $CS(3, K_4^{(3)} + e, 5k + r - 5)$ of type $(5^{k-1} : r)$ for $k = 4, 6$ are from Lemmas 3.4 and 3.5, and the needed $GDD(3, K_4^{(3)} + e, 5k)$ of type 5^k for $k = 4, 6$ are from Lemma 3.3. By Lemmas 3.7 and 3.8, there exist an $HS(3, K_4^{(3)} + e; 15, 5)$ and an $HS(3, K_4^{(3)} + e; 16, 6)$. Thus apply Construction 2.7 to obtain an $S(3, K_4^{(3)} + e, 10u + 5 + r)$. \square

Lemma 4.4 *There exists an $S(3, K_4^{(3)} + e, v)$ for $v \equiv 7 \pmod{10}$.*

Proof By Lemma 2.4, there exists an $S(3, \{4, 6\}, u + 1)$ (X, \mathcal{B}) with $u \equiv 1 \pmod{2}$ and $u \geq 3$. Then for any $x_0 \in X$, $(X, \{x_0\}, \{\{x\} : x \in X \setminus \{x_0\}\}, \mathcal{B})$ is a $CS(3, \{4, 6\}, u + 1)$ of type $(1^u : 1)$. Applying Construction 2.8 with $b = 5$ and $r = 2$, we have a $CS(3, K_4^{(3)} + e, 5u + 2)$ of type $(5^u : 2)$, where the needed $CS(3, K_4^{(3)} + e, 5k - 3)$ of type $(5^{k-1} : 2)$ for $k = 4, 6$ are from Lemma 3.6, and the needed $GDD(3, K_4^{(3)} + e, 5k)$ of type 5^k for $k = 4, 6$ are from Lemma 3.3. By Lemmas 3.1 there exists an $S(3, K_4^{(3)} + e, 7)$. Thus apply Construction 2.7 to obtain an $S(3, K_4^{(3)} + e, 5u + 2)$. \square

Proof of Theorem 1.4: The necessity follows from Lemmas 1.1 and 4.1. The sufficiency follows from Lemmas 4.2-4.4. \square

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